

SEIBERG-WITTEN EQUATIONS ON SURFACES OF LOGARITHMIC GENERAL TYPE

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ABSTRACT. We study the Seiberg-Witten equations on surfaces of logarithmic general type. First, we show how to construct irreducible solutions of the Seiberg-Witten equations for any metric which is “asymptotic” to a Poincaré type metric at infinity. Then we compute a lower bound for the L^2 -norm of scalar curvature on these spaces and give non-existence results for Einstein metrics on blow-ups.

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1. INTRODUCTION

This paper is concerned with *Seiberg-Witten equations* on non-compact complex 4-manifolds. More precisely, we are interested in complex manifolds obtained from a smooth projective surface \overline{M} by removing a smooth, reduced, not necessarily connected, divisor Σ . Recall that in this case a reduced divisor is simply a collection of complex submanifolds or otherwise said smooth holomorphic curves. We denote by M the complex manifold obtained from \overline{M} by removing Σ .

The main problem with Seiberg-Witten theory on non-compact manifold is the lack of a satisfactory existence theory. For a readable account of what is known in the compact case one may refer to [25].

Following a beautiful approach due to Biquard [5], we solve the SW equations on M by working on the compactification \overline{M} . More precisely, we produce an *irreducible* solution of the unperturbed SW equations on M as limit of solutions of the perturbed SW equations on \overline{M} . From the metric point of view, starting with (M, g) where g is assumed to be asymptotic to a Poincaré type metric at infinity, one has to construct a sequence (\overline{M}, g_j) of metric compactifications that approximate (M, g) as j diverges. The irreducible solution of the SW equations on (M, g) is then constructed by a bootstrap argument with the solutions of the SW equations on (\overline{M}, g_j) with suitably constructed perturbations.

Here is an outline of the paper. Section 5 describes explicitly the metric compactifications (\overline{M}, g_j) . These metrics are completely analogous to the one used by Rollin and Biquard in [27] and [5]. Furthermore, few results concerning the Riemannian geometry of the spaces (\overline{M}, g_j) are given.

In Section 4 we review some classical facts about the L^2 -cohomology of complete manifolds. Moreover, we recall a fundamental result of Zucker [31] concerning the L^2 -cohomology in the Poincaré metric. Finally, we formulate the L^2 -analogue of LeBrun's scalar curvature estimate [19].

Sections 6 and 7 contain the uniform Poincaré inequalities on functions and 1-forms needed for the main analytical argument. Moreover the convergence, as j goes to infinity, of the harmonic forms on (\overline{M}, g_j) is studied in detail.

In Section 8 the bootstrap argument is worked out. The existence result so obtained is summarized in Theorem A.

In Section 9, Theorem A is applied to derive several geometrical consequences. First, we give a lower bound for the Riemannian functional $\int s_g^2 d\mu_g$ on M , where by s we denote the scalar curvature. Second, an obstruction to the existence of Einstein metrics on blow-ups of M is given. These results are summarized in Theorem B and Theorem C. These theorems are the finite volume generalization of some well-known results of LeBrun for closed four manifolds, see for example [22].

2. LOGARITHMIC KODAIRA DIMENSION

In this section, we review some of the complex geometry needed in this paper. In algebraic geometry [15], the object of study is usually not an open complex manifold M but rather a *pair* (\overline{M}, Σ) consisting of an algebraic variety \overline{M} , and a reduced simple normal crossing divisor Σ . The obvious requirement is that M is biholomorphic to $\overline{M} \setminus \Sigma$. The

variety \overline{M} is called the *smooth completion* or simply the *compactification* of M . The divisor Σ is referred as the *boundary* of the smooth completion.

In this paper, unless otherwise stated, \overline{M} will be a smooth, projective, surface and Σ a reduced, smooth, divisor without rational components. The general case including rational components and simple normal crossing boundary divisors will be treated elsewhere.

Motivated by the well-known properties of the minimal model in complex dimension two [3], we introduce a notion of minimality for a pair (\overline{M}, Σ) . For more details see [15].

Definition 1. *Let (\overline{M}, Σ) be as above. The pair is called minimal if \overline{M} does not contain an exceptional curve E of the first kind such that $E \cdot \Sigma \leq 1$.*

Recall that an exceptional curve of the first kind is simply a rational curve E with self-intersection minus one, see [12] and [3].

Given a pair (\overline{M}, Σ) , consider the *logarithmic canonical bundle* $\mathcal{L} = K_{\overline{M}} + \Sigma$. Given any integer m , define the *logarithmic plurigenera* as $\overline{P}_m = \dim H^0(\overline{M}, \mathcal{O}(m\mathcal{L}))$. If $\overline{P}_m > 0$, we define the *m-th logarithmic canonical map* $\Phi_{m\mathcal{L}}$ of the pair (\overline{M}, Σ) by

$$\Phi_{m\mathcal{L}}(x) = [s_1(x), \dots, s_N(x)],$$

where s_1, \dots, s_N is a basis for the vector space $H^0(\overline{M}, \mathcal{O}(m\mathcal{L}))$. Here the point x has to be chosen not in the *base locus* B_m of the linear series $|m\mathcal{L}|$. Concretely, B_m is simply the set of points where all the sections of $m\mathcal{L}$ vanish. At this point one introduces the notion of *logarithmic Kodaira dimension* exactly as in the closed smooth case. More precisely, the logarithmic Kodaira dimension is defined to be the maximal dimension of the image of the logarithmic canonical maps. Conventionally, we assign dimension $-\infty$ to pairs for which the logarithmic plurigenera are identically zero. We denote this numerical invariant by $\overline{k}(M)$. A pair (\overline{M}, Σ) is said to be of *logarithmic general type* if it has maximal logarithmic Kodaira dimension.

Next, we have to recall the concept of *numerical effectiveness* for line bundles on surfaces. For an extensive treatment of this circle of ideas in complex algebraic geometry we refer to the book [17].

Definition 2. *Let \overline{M} be a smooth projective surface. A line bundle L on \overline{M} is said to be numerical effective, or simply nef, if*

$$\int_D c_1(L) \geq 0$$

for any irreducible curve D in \overline{M} .

We are now ready to state and prove the following result:

Lemma 2.1. *Let (\overline{M}, Σ) be a minimal pair such that Σ is a reduced, smooth divisor without rational curves. If $\overline{k}(M) \geq 0$, then \mathcal{L} is numerically effective.*

Proof. Let E be an irreducible components of the boundary divisor Σ . Since Σ does not contain rational components, by the adjunction formula we conclude that $\mathcal{L} \cdot E \geq 0$. Thus, it remains to check that $\mathcal{L} \cdot E \geq 0$ for any irreducible curve E which is not a component of Σ . Let us proceed by contradiction. Suppose $\mathcal{L} \cdot E < 0$. Since $\overline{k}(M) \geq 0$, there exists m such that $m\mathcal{L}$ is effective. We then conclude that E must be a component of $m\mathcal{L}$ with negative self-intersection. On the other hand

$$K_{\overline{M}} \cdot E + \Sigma \cdot E < 0 \implies K_{\overline{M}} \cdot E < 0.$$

It then follows that $E \simeq \mathbb{C}P^1$ and $E^2 = -1$. As a result we must have $\Sigma \cdot E = 0$. This contradicts the minimality of the pair. \square

We can now concentrate on the case when the pair (\overline{M}, Σ) is of logarithmic general type.

Proposition 2.2. *Let (\overline{M}, Σ) be a minimal pair such that Σ is a reduced, smooth divisor without rational curves. If $\overline{k}(M) = 2$, then $\mathcal{L}^2 > 0$.*

Proof. By Lemma 2.1, the line bundle \mathcal{L} is nef. Since the Kodaira dimension of \mathcal{L} is nonnegative, by Serre duality we conclude that $H^2(\overline{M}, \mathcal{O}(m\mathcal{L}))$ vanishes for all positive integers m . A Hirzebruch-Riemann-Roch computation shows that

$$\overline{P}_m = h^1(m\mathcal{L}) + \frac{m(m-1)}{2}\mathcal{L}^2 + \frac{m}{2}\mathcal{L} \cdot \Sigma + \chi_{\mathcal{O}}$$

where by $\chi_{\mathcal{O}}$ we denote the holomorphic Euler characteristic of \overline{M} . Now, by an important generalization of the classical Kodaira-Nakano vanishing theorem due to Kawamata and Viehweg [17], we know that $h^1(m\mathcal{L}) = 0$ for any positive m . Since \overline{P}_m grows quadratically in m , we conclude that $\mathcal{L}^2 > 0$. \square

We conclude this section with a proposition regarding the *ampleness* properties of \mathcal{L} . This proposition will be of some importance in Section 9.

Proposition 2.3. *Let (\overline{M}, Σ) be a minimal pair such that Σ is smooth and reduced. The log-canonical bundle \mathcal{L} is ample iff $\overline{k}(M) = 2$, Σ*

does not contain rational and/or elliptic components, and there are no interior rational (-2) -curves.

Proof. If \mathcal{L} is ample, we clearly have $\bar{k}(M) = 2$. Moreover by the adjunction formula Σ cannot contain rational and/or elliptic components, and there are no interior rational (-2) -curves, i.e., (-2) -curves not intersecting the boundary divisor Σ . Assume now (\bar{M}, Σ) to be log-general and with Σ not containing rational and elliptic curves. By Proposition 2.2 we know that $\mathcal{L}^2 > 0$. By the Hodge index theorem any divisor E such that $\mathcal{L} \cdot E = 0$ must have negative self-intersection. Since by assumption the pair is minimal and there are no interior (-2) -curves, the proposition is now a consequence of Nakai's criterion for ampleness of divisors on surfaces, see [3]. \square

3. POINCARÉ METRICS

Let (\bar{M}, ω_0) be a smooth algebraic surface equipped with a Kähler metric. Let Σ be a smooth, not necessarily connected, reduced divisor on \bar{M} . Choose an Hermitian metric $\|\cdot\|$ on $\mathcal{O}_{\bar{M}}(\Sigma)$. Let $s \in H^0(\bar{M}, \mathcal{O}(\Sigma))$ be a defining section for the divisor Σ . On M , for $T > 0$ big enough

$$(1) \quad \omega = T\omega_0 - \sqrt{-1}\partial\bar{\partial} \log \log^2 \|s\|^2$$

defines a complete cuspidal Kähler metric. On each cusp this metric asymptotically looks like a product of the Poincaré punctured metric on the disk and the divisor. The number of cusp is clearly in one to one correspondence with the number of connected components of Σ which we denote by Σ_i . More precisely, one can easily show that on each cusp the Riemannian metric associated to (1) is given by

$$(2) \quad g = dt^2 + e^{-2t}\eta_i^2 + p^*g_{\Sigma_i} + O(e^{-t})$$

where η_i is a connection 1-form for the normal bundle of Σ_i , g_{Σ_i} is a *fixed* metric on Σ_i , and where the perturbation $O(e^{-t})$ is understood at any order for g . Note that the metric g_{Σ_i} is simply obtained by restricting

$$\hat{\omega} = T\omega_0 - \frac{2\sqrt{-1}}{\log \|s\|^2} \partial\bar{\partial} \log \|s\|^2$$

to the holomorphic curve Σ_i . Finally, we denote by

$$p : N_{\Sigma_i} \longrightarrow \Sigma_i$$

the S^1 bundle associated to the normal bundle of Σ_i in \overline{M} . From the Riemannian geometry point of view we have

$$g = dt^2 + g_t + O(e^{-t})$$

where g_t is a 1-parameter families of metrics on N_{Σ_i} which are *Riemannian submersions* with respect to p_{η_i} and g_{Σ_i} . These metrics are *volume collapsing* in the direction transverse to the divisor Σ only.

4. L^2 -COHOMOLOGY AND SEIBERG-WITTEN ESTIMATES

Let us begin with a review of some facts about L^2 -cohomology and its relation to the space of L^2 -harmonic forms. The interested reader is referred to [1] for a nice survey in this area. Given a orientable non-compact manifold (M, g) we have, when the differential d is restricted to an appropriate dense subset, a Hilbert complex

$$\dots \longrightarrow L^2\Omega_g^{k-1}(M) \longrightarrow L^2\Omega_g^k(M) \longrightarrow L^2\Omega_g^{k+1}(M) \longrightarrow \dots$$

where the inner products on the exterior bundles are induced by g . Define the maximal domain of d , at the k -th level, to be

$$Dom^k(d) = \{\alpha \in L^2\Omega_g^k(M), d\alpha \in L^2\Omega_g^{k+1}(M)\}$$

where $d\alpha \in L^2\Omega_g^{k+1}(M)$ is to be understood in the distributional sense. The (reduced) L^2 -cohomology groups are then defined to be

$$H_2^k(M) = Z_2^k(M) / \overline{dDom^{k-1}(d)},$$

where

$$Z_2^k(M) = \{\alpha \in L^2\Omega_g^k(M), d\alpha = 0\}.$$

On (M, g) there is a Hodge-Kodaira decomposition

$$L^2\Omega_g^k(M) = \mathcal{H}_g^k(M) \oplus \overline{dC_c^\infty\Omega^{k-1}} \oplus \overline{d^*C_c^\infty\Omega^{k+1}},$$

where

$$\mathcal{H}_g^k(M) = \{\alpha \in L^2\Omega_g^k(M), d\alpha = 0, d^*\alpha = 0\}.$$

Moreover, if we assume (M, g) to be complete the maximal and minimal domain of d coincide. In other words

$$\overline{dDom^{k-1}(d)} = \overline{dC_c^\infty\Omega^{k-1}},$$

which implies

$$H_2^k(M) = \mathcal{H}_g^k(M).$$

Summarizing, if the manifold is complete, the harmonic L^2 -forms compute the reduced L^2 -cohomology. Moreover, in this case the L^2 -harmonic forms can be characterized as follows

$$\mathcal{H}_g^k(M) = \{\alpha \in L^2\Omega_2^k(M), (dd^* + d^*d)\alpha = 0\}.$$

Finally, the orientability of M gives a duality isomorphism via the Hodge $*$ operator

$$\mathcal{H}_g^k(M) \simeq \mathcal{H}_g^{n-k}(M).$$

If the manifold M has dimension $4n$ it then makes sense to talk about L^2 self-dual and anti-self-dual forms on $L^2\Omega_g^{2n}(M)$. If $\mathcal{H}_g^{2n}(M)$ is finite dimensional, the concept of L^2 -signature is well defined.

Apart from the basic concepts above, what will be important for us is the following theorem of Zucker [31]. This classical result provides a complete topological interpretation of the L^2 -cohomology groups of finite volume complex manifolds equipped with Poincaré type metrics. The statement presented here is adapted from [31] to fit with the purpose and notation of this work.

Theorem 1 (Zucker). *Let \overline{M} be a smooth algebraic manifold. Let Σ be a reduced, smooth, divisor. If M is equipped with a metric g quasi-isometric to a Poincaré type metric then*

$$H_2^*(M) \simeq H^*(\overline{M}; \mathbb{R}).$$

We close this section with the L^2 -analogue of some well-known Seiberg-Witten estimates due to LeBrun [19]. Let (M, g) be a complete finite-volume 4-manifold. Let \mathcal{L} be a complex line bundle on M . By extending the Chern-Weil theory for compact manifolds, we can define the L^2 -Chern class of \mathcal{L} . More precisely, given a connection A on \mathcal{L} such that $F_A \in L^2\Omega_g^2(M)$, we may define

$$c_1(\mathcal{L}) = \frac{i}{2\pi}[F_A]_{L^2}$$

where with F_A we indicate the curvature of the given connection. It is an interesting corollary of the L^2 -cohomology theory that, on complete manifolds, such an L^2 -cohomology element is connection independent as long as we allow connections that differ by a 1-form in the maximal domain of the d operator. More precisely, let A' be a connection on \mathcal{L} such that $A' = A + \alpha$ with $\alpha \in L_1^2\Omega_g^1(M)$. We then have $F_{A'} = F_A + d\alpha$ and therefore by the Hodge-Kodaira decomposition we conclude that $\frac{i}{2\pi}[F_A]_{L^2} = \frac{i}{2\pi}[F_{A'}]_{L^2}$. Similarly, the associated L^2 -Chern number $c_1^2(\mathcal{L})$ is also well defined.

The following lemma is an easy consequence of the Hodge-Kodaira decomposition. For the details of the proof see [8].

Lemma 4.1. *Given \mathcal{L} and A as above, we have*

$$\int_M |F_A^+|^2 d\mu_g \geq 4\pi^2 (c_1^+(\mathcal{L}))^2$$

where $c_1^+(\mathcal{L})$ is the self-dual part of the g -harmonic L^2 -representative of $[c_1(\mathcal{L})]$.

We can now formulate the L^2 -analogue of the scalar curvature estimate discovered in [19] for compact manifolds.

Theorem 4.2. *Let (M^4, g) be a finite volume Riemannian manifold where g is C^2 -asymptotic to a Poincaré metric. Let $(A, \psi) \in L_1^2(M, g)$ be an irreducible solution of the SW equations associated to a Spin^c structure \mathfrak{c} with determinant line bundle \mathcal{L} . Then*

$$\int_M s_g^2 d\mu_g \geq 32\pi^2 (c_1^+(\mathcal{L}))^2$$

with equality if and only if g has constant negative scalar curvature, and is Kähler with respect to a complex structure compatible with \mathfrak{c} .

Proof. The proof is largely based on an idea of LeBrun [19]. By using Lemma 4.1, the proof reduces to an integration by part using the completeness of g . For the analytical details one may refer to [27] and [8]. \square

5. THE METRIC COMPACTIFICATIONS

Given (M, g) , where g is a Poincaré metric, we want to briefly describe a family of metric compactifications (\overline{M}, g_j) . Clearly, each of the cusp end of M can be closed topologically as a manifold by adding the corresponding divisor Σ_i . Recall that on each cusp end of M the metric g is given, in appropriate coordinates, by $g = dt^2 + e^{-2t}\eta^2 + g_{\Sigma_i} + O(e^{-t})$ and for $t > 0$. We can then define, on a closed tubular neighborhood of Σ in \overline{M} , a sequence of metrics $\{\tilde{g}_j\}$ by

$$\tilde{g}_j = dt^2 + \varphi_j^2(t)\eta_i^2 + g_{\Sigma_i}$$

where the index i runs over all the connected components of Σ and where $\varphi_j(t)$ is a smooth warping function such that:

- (1) $\varphi_j(t) = e^{-t}$ for $t \in [0, j+1]$;
- (2) $\varphi_j(t) = T_j - t$ for $t \in [j+1+\epsilon, T_j]$.

Here ϵ is a fixed number that can be chosen to be small, and T_j is an appropriate number bigger than $j + 1 + \epsilon$. Because of the second condition above, \tilde{g}_j is smooth for t approaching T_j . For later convenience we want to prescribe in more details the behavior of $\varphi_j(t)$ in the interval $t \in [j + 1, j + 1 + \epsilon]$. We require that $\partial_t^2 \varphi_j(t)$ decreases from $e^{-(j+1)}$ to 0 in the interval $[j + 1, j + 1 + \delta_j]$ where δ_j is a positive number less than ϵ . Then for $t \in [j + 1 + \delta_j, \epsilon]$, we make $\partial_t^2 \varphi_j$ very negative in order to decrease $\partial_t \varphi_j$ to -1 and smoothly glue $\varphi_j(t)$ to the function $T_j - t$. Moreover, by eventually letting the parameters δ_j go to zero as j goes to infinity, we require $\frac{|\partial_t \varphi_j|}{\varphi_j}$ to be increasing in the interval $[j + 1, j + 1 + \delta_j]$. Finally, we require $\frac{|\partial_t \varphi_j|}{\varphi_j}$ to be bounded from above uniformly in j . These conditions on the warping factors φ_j are taken from [27]. They are particularly useful in proving the uniform Poincaré inequality on 1-form given in section 6.

With the metrics \tilde{g}_j at our disposal, we are now ready to describe the family of metric compactifications (\overline{M}, g_j) . For later convenience, we allow the metric g to be asymptotic to a Poincaré type metric in the C^2 -topology. Thus, if g is such a metric we set

$$g_j = (1 - \chi_j)g + \chi_j \tilde{g}_j$$

where $\chi_j(t)$ is a sequence of smooth increasing functions defined on the cusps of M such that $\chi_j(t) = 0$ if $t \leq j$ and $\chi_j(t) = 1$ if $t \geq j + 1$. Note that the metrics g_j are by construction isometric to g on bigger and bigger compact sets of M as j goes to infinity. In this sense we think the sequence of Riemannian manifolds (\overline{M}, g_j) as compact approximations of (M, g) .

We conclude this section with two simple propositions regarding the volume and the scalar curvature of the Riemannian manifolds (\overline{M}, g_j) .

Proposition 5.1. *The scalar curvature of the metrics $\{g_j\}$ can be expressed as*

$$s_{g_j} = s_{g_j}^b - 2\chi_j \frac{\partial_t^2 \varphi_j}{\varphi_j}$$

where $s_{g_j}^b$ is a smooth function on \overline{M} that can be bounded uniformly in j .

Proposition 5.2. *There exists a constant $K > 0$ such that*

$$\text{Vol}_{g_j}(\overline{M}) \leq K$$

for any j .

One may refer to [9] and [27] for more details.

6. POINCARÉ INEQUALITIES AND CONVERGENCE OF 1-FORMS

We need to show that, given the sequence of metrics $\{g_j\}$, we can find a uniform Poincaré inequality on functions. We have the following lemma.

Lemma 6.1. *Consider the metric $g = dt^2 + g_t$ on the product $[0, \infty) \times N$, such that the mean curvature of the cross section N is uniformly bounded from below by a positive constant h_0 . Then, for any function f we have*

$$\int |\partial_t f|^2 d\mu_g \geq h_0^2 \int |f|^2 d\mu_g + h_0 \int_{t=T} |f|^2 d\mu_{g_t} - h_0 \int_{t=0} |f|^2 d\mu_{g_t}.$$

Proof. See Lemma 4.1 in [5]. \square

By definition of the metrics $\{g_j\}$, the mean curvature of the cross sections N_t on the cusps are uniformly bounded from below independently of j . Using Lemma 6.1, we can now derive the desired uniform Poincaré inequality on functions.

Proposition 6.2. *There exists a positive constant c , independent of j , such that*

$$\int_{\overline{M}} |df|^2 d\mu_{g_j} \geq c \int_{\overline{M}} |f|^2 d\mu_{g_j}$$

for any function f on \overline{M} such that $\int_{\overline{M}} f d\mu_{g_j} = 0$.

Proof. The argument is by contradiction. The details can be found in [5], see Corollaire 4.3. \square

Next, we have to derive an uniform Poincaré inequality for 1-forms. Given a 1-form α the following lemma holds:

Lemma 6.3. *There exists $T > 0$ such that*

$$\int_N |\nabla \alpha|^2 + Ric^{g_j}(\alpha, \alpha) d\mu_{g_t} \geq k_1 \int_N |\nabla_{\partial_t} \alpha|^2 d\mu_{g_t} - k_2 \int_N |\alpha|^2 d\mu_{g_t}$$

for any $t \in [T, T_j)$, with $k_1 > 0$ and $k_2 = O(e^{-t})$.

Proof. Let us start by considering a Poncaré metric g on M and the associated sequence of metrics $\{g_j\}$. On each cusp, write α as

$$\alpha = f dt + \alpha_1$$

where $i_{\partial_t} \alpha_1 = 0$, with f a real functions. We then have

$$\nabla \alpha = dt \otimes \nabla_{\partial_t} \alpha + d^N f \otimes dt - f \Pi_{g_j}(\cdot, \cdot) + \Pi_{g_j}(\alpha_1, \cdot) \otimes dt + \nabla^N \alpha_1$$

where by II we denote the second fundamental form of the slice N . By further decomposing α_1 as

$$\alpha_1 = f_1 \varphi_j d\theta + \alpha_2$$

with f_1 a real function and where $i_{v_\theta} \alpha_2 = 0$, we can compute all the components of $\nabla \alpha$. Next, one computes $Ric^{g_j}(\alpha, \alpha)$. More precisely, we have

$$Ric^{g_j}(\alpha, \alpha) = Ric^{g_{\Sigma_i}}(\alpha_2, \alpha_2) - \frac{\partial_t^2 \varphi_j}{\varphi_j} \{|f|^2 + |f_1|^2\} + O(e^{-2t})|\alpha|^2.$$

where $Ric^{g_{\Sigma_i}}$ is the Ricci curvature of the metric g_{Σ_i} coming from 2. Thus, following a strategy first outlined by Rollin in [27] and further employed in [9] and [10], the lemma is a consequence of the Poincaré inequality for the circle. \square

The idea is now to integrate with respect to the t variable on each cusp. This will allow us to globalize the slice estimate given in Lemma 6.3. First, observe that for $[t_1, t_2] \subset [T, T_j]$

$$\begin{aligned} \int_{\partial\{[t_1, t_2] \times N\}} |\alpha|^2 d\mu_{g_j} &= \int_{[t_1, t_2] \times N} \partial_t(|\alpha|^2 d\mu_{g_t}) dt = \int_{[t_1, t_2] \times N} \partial_t |\alpha|^2 d\mu_{g_t} dt \\ &\quad + \int_{[t_1, t_2] \times N} |\alpha|^2 \partial_t d\mu_{g_t} dt \\ &= \int_{[t_1, t_2] \times N} \partial_t |\alpha|^2 d\mu_{g_j} - 2 \int_{[t_1, t_2] \times N} h |\alpha|^2 d\mu_{g_j}. \end{aligned}$$

We then obtain

$$\int_{[t_1, t_2] \times N} \partial_t |\alpha|^2 d\mu_{g_j} \geq \int_{\partial\{[t_1, t_2] \times N\}} |\alpha|^2 d\mu_{g_j} + 2h_0 \int_{[t_1, t_2] \times N} |\alpha|^2 d\mu_{g_j}.$$

where h_0 is a uniform lower bound for the mean curvature. But now

$$\partial_t |\alpha|^2 = 2(\alpha, \nabla_{\partial_t} \alpha) \leq 2|\alpha| |\nabla_{\partial_t} \alpha| \leq h_0 |\alpha|^2 + \frac{1}{h_0} |\nabla_{\partial_t} \alpha|^2$$

which then implies

$$(3) \quad \int_{[t_1, t_2] \times N} |\nabla_{\partial_t} \alpha|^2 d\mu_{g_j} \geq h_0 \int_{\partial\{[t_1, t_2] \times N\}} |\alpha|^2 d\mu_{g_j} + h_0^2 \int_{[t_1, t_2] \times N} |\alpha|^2 d\mu_{g_j}.$$

We summarize the discussion above into the following lemma.

Lemma 6.4. *There exist positive numbers $c > 0$, $T > 0$ such that*

$$\int_{[t_1, t_2] \times N} |d\alpha|^2 + |d^{*g_j} \alpha|^2 d\mu_{g_j} \geq c \int_{[t_1, t_2] \times N} |\alpha|^2 d\mu_{g_j}$$

for any $[t_1, t_2] \subset [T, T_j]$ and α with support contained in $[t_1, t_2] \times N$.

Proof. Combining (3) and Lemma 6.3 with T big enough, the result follows from the well know Bochner formula for 1-forms. \square

The above lemma is almost the desired uniform Poincaré inequality. To conclude the proof we need few results concerning the convergence of harmonic 1-forms.

Proposition 6.5. *Let $[a] \in H_{dR}^1(\overline{M})$ and $\{\alpha_j\}$ be the sequence of harmonic representatives with respect the metrics $\{g_j\}$. Then $\{\alpha_j\}$ converges, with respect to the C^∞ -topology on compact sets, to a harmonic 1-form $\alpha \in L^2\Omega_g^1(M)$.*

Proof. Let β be a closed smooth representative for $[a] \in H_{dR}^1(\overline{M})$. Given g_j , by the Hodge decomposition theorem, we can write $\alpha_j = \beta + df_j$ with α_j harmonic and f_j a C^∞ function. Without loss of generality we can assume that $\int_{\overline{M}} f_j d\mu_j = 0$. Furthermore, we have

$$0 = d^*\alpha_j = d^*\beta + d^*df_j \implies \Delta_H^{g_j} f_j = -d^*\beta$$

where by Δ_H we denote the Hodge Laplacian. Finally, a bootstrap argument with elliptic equation above combined with Proposition 6.2 gives the convergence result. For more details see Proposition 4.4. in [5]. \square

It is now possible to refine Proposition 6.5 and analyze the convergence in more details. Let β be a smooth representative of the cohomology class $[a] \in H_{dR}^1(\overline{M})$. By the long exact sequence with compact support of the pair (\overline{M}, Σ)

$$\dots \longrightarrow H_c^1(M) \longrightarrow H^1(\overline{M}) \longrightarrow H^1(\Sigma) \longrightarrow \dots$$

we can chose β as follows

$$\beta = \beta_c + \sum_i \gamma_i$$

where β_c is a smooth closed 1-form with support not intersecting the divisor Σ and $\gamma_i \in H^1(\Sigma_i; \mathbb{R})$ for any i . The metric g is C^2 -asymptotic to a Poincaré metric, as a result

$$\lim_{t \rightarrow \infty} d^{*g} \gamma_i = 0$$

since γ_i can be chosen to be harmonic with respect to the metric g_{Σ_i} for any choice of the index i . Furthermore, given $\epsilon > 0$ we can find T big enough such that $\lim_{j \rightarrow \infty} \|d^{*g_j} \gamma_i\|_{L_{g_j}^2(t \geq T)} \leq \epsilon$. In other words we proved

Lemma 6.6. *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t \geq T} |d^* \beta|^2 d\mu_g \leq \epsilon, \quad \int_{t \geq T} |d^{*j} \beta|^2 d\mu_{g_j} \leq \epsilon.$$

We can now prove

Lemma 6.7. *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t \geq T} |\alpha|^2 d\mu_g \leq \epsilon, \quad \int_{t \geq T} |\alpha_j|^2 d\mu_{g_j} \leq \epsilon.$$

Proof. By construction $\alpha_j = \beta + df_j$, thus

$$\int_{t \geq T} |df_j|^2 d\mu_{g_j} = \int_{t=T} f_j \wedge *df_j - \int_{t \geq T} (d^* df_j, f_j) d\mu_{g_j}.$$

But now

$$d^{*j} \alpha_j = d^{*j} \beta + d^{*j} df_j = 0 \implies d^{*j} df_j = -d^{*j} \beta,$$

thus

$$\int_{t \geq T} |df_j|^2 d\mu_{g_j} = \int_{t=T} f_j \wedge *df_j + \int_{t \geq T} (d^* \beta, f_j) d\mu_{g_j}.$$

By the Cauchy inequality

$$\int_{t \geq T} (d^* \beta, f_j) d\mu_{g_j} \leq \|f_j\|_{L_{g_j}^2} \|d^{*j} \beta\|_{L_{g_j}^2(t \geq T)}$$

and then this term can be made arbitrarily small. It remains to study the term $\int_{t=T} f_j \wedge *df_j$. Recall that $f_j \rightarrow f$ in the C^∞ topology on compact sets. Thus, for a fixed T

$$\int_{t=T} f_j \wedge *df_j \rightarrow \int_{t=T} f \wedge *df.$$

It remains to show that $\int_{t=T} f \wedge *df$ can be made arbitrarily small by taking T big enough. Define the function $F(s) = \int_{t=s} f * df$, since $f \in L_1^2$ we have $F(s) \in L^1(\mathbb{R}^+)$ and then we can find a sequence $\{s_k\} \rightarrow \infty$ such that $F(s_k) \rightarrow 0$.

□

Proposition 6.8. *There exists $c > 0$ independent of j such that*

$$\int_M |d\alpha|^2 + |d^{*g_j} \alpha|^2 d\mu_{g_j} \geq c \int_M |\alpha|^2 d\mu_{g_j}$$

for any $\alpha \perp \mathcal{H}_{g_j}^1$.

Proof. Let us proceed by contradiction. Assume the existence of a sequence $\{\alpha_j\} \in (\mathcal{H}_{g_j}^1)^\perp$ such that $\|\alpha_j\|_{L^2(g_j)} = 1$ and for which

$$\int_{\overline{M}} |d\alpha_j|^2 + |d^{*g_j}\alpha_j|^2 d\mu_{g_j} \longrightarrow 0$$

as $j \rightarrow \infty$. By eventually passing to a subsequence, a diagonal argument shows that $\{\alpha_j\}$ converges, with respect to the C^∞ -topology on compact sets, to a 1-form $\alpha \in L^2\Omega_g^1(M)$. By construction $\alpha \in \mathcal{H}_g^1(M)$. On the other hand, Lemma 6.7 combined with the isomorphism provided by Theorem 1 gives that $\alpha \in (\mathcal{H}_g^1)^\perp$. We conclude that $\alpha = 0$. Lemma 6.4 can now be easily applied to derive a contradiction. \square

7. CONVERGENCE OF 2-FORMS

In this section we have to study the convergence of 2-forms. The first result is completely analogous to the case of 1-forms.

Proposition 7.1. *Let $[a] \in H_{dR}^2(\overline{M})$ and $\{\alpha_j\}$ be the sequence of harmonic representatives with respect the sequence of metrics $\{g_j\}$. Then $\{\alpha_j\}$ converges, with respect to the C^∞ -topology on compact sets, to a harmonic 2-forms $\alpha \in L^2\Omega_g^2(M)$.*

Proof. Given an element $a \in H_{dR}^2(\overline{M})$, take a smooth representative of the form $\beta = \beta_c + \sum_i \gamma_i$ where β_c is a closed 2-form with support not intersecting Σ and $\gamma_i \in H^2(\Sigma_i; \mathbb{R})$ for any i . Given g_j , let α_j be the harmonic representative of the cohomology class determined by a . By the Hodge decomposition theorem we can write $\alpha_j = \beta + d\sigma_j$ with $\sigma_j \in (\mathcal{H}_{g_j}^1)^\perp$ such that $d^{*j}\sigma_j = 0$. Thus

$$0 = d^{*j}\beta + d^{*j}d\sigma_j \implies d^{*j}d\sigma_j = -d^{*j}\beta.$$

Taking the global L^2 inner product of $d^{*j}d\sigma_j$ with σ_j we obtain the estimate

$$\begin{aligned} (4) \quad (d^{*j}d\sigma_j, \sigma_j)_{L^2(g_j)} &= \|d\sigma_j\|_{L^2}^2 = - \int_{\overline{M}} (\sigma_j, d^{*j}\beta) d\mu_{g_j} \\ &\leq \|\sigma_j\|_{L^2(g_j)} \|d^{*j}\beta\|_{L^2(g_j)}. \end{aligned}$$

By Proposition 6.8, we conclude that

$$(5) \quad \|\sigma_j\|_{L^2(g_j)}^2 \leq c \|d\sigma_j\|_{L^2(g_j)}^2.$$

Combining 4 and 5 we then obtain

$$\|\sigma_j\|_{L^2(g_j)}^2 \leq c \|d\sigma_j\|_{L^2(g_j)}^2 \leq c \|\sigma_j\|_{L^2(g_j)} \|d^{*j}\beta\|_{L^2(g_j)}.$$

Since $\|d^{*j}\beta\|_{L^2(g_j)}$ is bounded independently of j , we conclude that the same is true for $\|\sigma_j\|_{L^2(g_j)}$ and $\|d\sigma_j\|_{L^2(g_j)}$. We conclude that $\|\sigma_j\|_{L^2_1(g_j)}$

is uniformly bounded. Now a standard diagonal argument allows us to conclude that, up to a subsequence, $\{\sigma_j\}$ weakly converges to an element $\sigma \in L_1^2$. Using the elliptic equation

$$\Delta_H^{g_j} \sigma_j = -d^{*j} \beta$$

and a bootstrapping argument it is possible to show that $\sigma_j \rightarrow \sigma$ in the C^∞ topology on compact sets. This proves the proposition. \square

We now want to obtain a refinement of Proposition 7.1. We begin with the following simple lemma.

Lemma 7.2. *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t \geq T} |d^{*g} \beta|^2 d\mu_g \leq \epsilon, \quad \int_{t \geq T} |d^{*j} \beta|^2 d\mu_{g_j} \leq \epsilon.$$

Proof. Since $\beta = \beta_c + \gamma$ with γ a fixed element in $H^2(\Sigma; \mathbb{R})$, the lemma follows from the definition of the metrics $\{g_j\}$. \square

An analogous result holds for the 2-forms $\{d\sigma_j\}$.

Lemma 7.3. *Given $\epsilon > 0$, there exists T big enough such that*

$$\int_{t \geq T} |d\sigma|^2 d\mu_g \leq \epsilon, \quad \int_{t \geq T} |d\sigma_j|^2 d\mu_{g_j} \leq \epsilon.$$

Proof. The first inequality follows easily from the fact that $\alpha \in L^2 \Omega_g^2(M)$. By Lemma 7.2, given $\epsilon > 0$ we can find T such that

$$\|\sigma_j\|_{L^2(g_j)} \left\{ \int_{t \geq T} |d^* \beta|^2 d\mu_{g_j} \right\}^{\frac{1}{2}} \leq \frac{\epsilon}{2}$$

independently of the index j . Now

$$\int_{t \geq T} |d\sigma_j|^2 d\mu_{g_j} = \int_{t=T} \sigma_j \wedge *_{g_j} d\sigma_j - \int_{t \geq T} (d^{*j} d\sigma_j, \sigma_j) d\mu_{g_j}$$

but $d^{*j} d\sigma_j = -d^{*j} \beta$, thus

$$\int_{t \geq T} |d\sigma_j|^2 d\mu_{g_j} \leq \frac{\epsilon}{2} + \left| \int_{t=T} \sigma_j \wedge *_{g_j} d\sigma_j \right|.$$

Since $\sigma_j \rightarrow \sigma$ in the C^∞ topology on compact sets, we have that $\int_{t=T} \sigma_j \wedge *_{g_j} d\sigma_j \rightarrow \int_{t=T} \sigma \wedge *_{g_j} d\sigma$. But now $\sigma \in L_1^2(g)$ and therefore we can conclude the proof of the proposition. \square

Lemma 7.4. *σ is orthogonal to the harmonic 1-form on (M, g) .*

Proof. By construction we have $\sigma_j \in (\mathcal{H}_{g_j}^1)^\perp$. Recall that fixed a cohomology element $[a] \in H_{dR}^1(\overline{M})$, denoted by $\{\gamma_j\}$ the sequence of the harmonic representatives with respect to the $\{g_j\}$, given $\epsilon > 0$ we can chose T such that $\int_{t \geq T} |\gamma_j|^2 d\mu_{g_j} \leq \epsilon$. Now, given $\gamma \in \mathcal{H}_g^1$ we want to show that $(\sigma, \gamma)_{L^2(g)} = 0$. Since $H_{dR}^1(\overline{M}) = \mathcal{H}_g^1(M)$, we can find a sequence of harmonic 1-forms $\{\gamma_j\}$ such that $\gamma_j \rightarrow \gamma$ in the C^∞ topology on compact sets. Let K be a compact set in M , then

$$(6) \quad \left| \int_{\overline{M} \setminus K} (\sigma_j, \gamma_j) d\mu_{g_j} \right| \leq \|\sigma_j\|_{L_{g_j}^2} \|\gamma_j\|_{L_{g_j}^2(\overline{M} \setminus K)}$$

can be made arbitrarily small by choosing the compact K big enough. Since $(\sigma_j, \gamma_j)_{L^2(\overline{M}, g_j)} = 0$, we have

$$\int_K (\sigma_j, \gamma_j) d\mu_{g_j} = - \int_{\overline{M} \setminus K} (\sigma_j, \gamma_j) d\mu_{g_j}$$

and then the integral $\int_K (\sigma_j, \gamma_j) d\mu_{g_j}$ can be made arbitrarily small. On the other hand

$$\left| \int_M (\sigma, \gamma) d\mu_g \right| \leq \left| \int_K (\sigma, \gamma) d\mu_g \right| + \|\sigma\|_{L^2(M, g)} \|\gamma\|_{L_g^2(M \setminus K)}.$$

Since $\gamma \in L^2 \Omega_g^1(M)$ we conclude that $\sigma \in (\mathcal{H}_g^1)^\perp$. \square

We now want to study the intersection form of (\overline{M}, g_j) and eventually show the convergence to the L^2 intersection form of (M, g) . Recall the isomorphism $H_{dR}^2(\overline{M}) \simeq \mathcal{H}^2(M)$, moreover given $[a] \in H_{dR}^2(\overline{M})$ we can generate $\{\alpha_j\} \in \mathcal{H}_{g_j}^2(\overline{M})$ that converges in the C^∞ topology on compact sets to a $\alpha \in \mathcal{H}_g^2(M)$. We also have that, fixed a compact set K , $*_j = *_g$ for j big enough. Since

$$\alpha_j = \alpha_j^{+j} + \alpha_j^{-j} = \frac{\alpha_j + *_j \alpha_j}{2} + \frac{\alpha_j - *_j \alpha_j}{2} \rightarrow \alpha^{+g} + \alpha^{-g} = \alpha$$

the claim follows. Let us summarize these results into a proposition.

Proposition 7.5. *Let (\overline{M}, g_j) and (M, g) be defined as above. Given $[a] \in H_{dR}^2(\overline{M})$ and denoted by $\{\alpha_j\}$ the harmonic representatives with respect to the sequence of metrics $\{g_j\}$, we have*

$$\|\alpha_j^+\|_{L^2(\overline{M}, g_j)} \rightarrow \|\alpha^+\|_{L^2(M, g)}, \quad \|\alpha_j^-\|_{L^2(\overline{M}, g_j)} \rightarrow \|\alpha^-\|_{L^2(M, g)}.$$

8. BIQUARD'S CONSTRUCTION

In this section we show how to construct an irreducible solution of the Seiberg-Witten equations on (M, g) , for *any* metric g which is C^2 -asymptotic to a Poincaré type metric at infinity.

Fix a $Spin^c$ structure on \overline{M} , with determinant line bundle L , and let g be a Poincaré metric on $\overline{M} \setminus \Sigma$. Let $\{g_j\}$ be the sequence of metrics on \overline{M} approximating (M, g) constructed in Section 5. Let us assume we can find irreducible solutions (A_j, ψ_j) of the perturbed Seiberg-Witten equations on (\overline{M}, g_j)

$$\begin{cases} \mathcal{D}_{A_j} \psi_j = 0 \\ F_{A_j}^+ + i2\pi\omega_j^+ = q(\psi_j) \end{cases}$$

where $\omega_j = \frac{i}{2\pi} F_{B_j}$ and B_j is the connection on the line bundle $\mathcal{O}_{\overline{M}}(\Sigma)$ given by

$$B_j = d - \sum_k i\chi_j(\partial_t \varphi_j) \eta_k.$$

The idea is to show that, up to gauge transformations, the (A_j, ψ_j) converge in the C^∞ topology on compact sets to a solution of the unperturbed Seiberg-Witten equations

$$\begin{cases} \mathcal{D}_A \psi = 0 \\ F_A^+ = q(\psi) \end{cases}$$

on (M, g) , where $A = C + a$, with C a fixed smooth connection on $L \otimes \mathcal{O}(-\Sigma)$, and $a \in L_1^2(\Omega_g^1(M))$ with $d^*a = 0$.

Lemma 8.1. *We have the decomposition*

$$\begin{aligned} s_{g_j} &= s_{g_j}^b - 2\chi_j \frac{\partial_t^2 \varphi_j}{\varphi_j} \\ F_{B_j} &= - \sum_k i\chi_j \frac{\partial_t^2 \varphi_j}{\varphi_j} dt \wedge \varphi_j \eta_k + F_j^b \end{aligned}$$

with $s_{g_j}^b$ and F_j^b bounded independently of j

Proof. See Proposition 5.1. □

Since $i2\pi\omega_j = -F_{B_j}$, we can rewrite the perturbed Seiberg-Witten equations as follows

$$(7) \quad \begin{cases} \mathcal{D}_{A_j} \psi_j = 0 \\ F_{A_j}^+ - F_{B_j}^+ = q(\psi_j). \end{cases}$$

Recall that in the case under consideration, the twisted Licherowicz formula [16] reads as follows

$$\mathcal{D}_{A_j}^2 \psi_j = \nabla_{A_j}^* \nabla_{A_j} \psi_j + \frac{s_{g_j}}{4} \psi_j + \frac{1}{2} F_{A_j}^+ \cdot \psi_j.$$

By using the SW equations we have

$$0 = \nabla_{A_j}^* \nabla_{A_j} \psi_j + \frac{s_{g_j}}{4} \psi_j + \frac{|\psi_j|^2}{4} \psi_j + \frac{1}{2} F_{B_j}^+ \cdot \psi_j.$$

Keeping into account the decomposition given in Lemma 8.1 we obtain

$$0 = \nabla_{A_j}^* \nabla_{A_j} \psi_j + P_j \psi_j + P_j^b \psi_j + \frac{|\psi_j|^2}{4} \psi_j$$

where on each cusp

$$P_j \psi_j = -\frac{1}{2} \chi_j \frac{\partial_t^2 \varphi_j}{\varphi_j} \psi_j - \frac{i}{2} \chi_j \frac{\partial_t^2 \varphi_j}{\varphi_j} (dt \wedge \varphi_j \eta_i)^+ \cdot \psi_j$$

with P_j^b uniformly bounded in j . Now, it can be explicitly checked that for a metric of the form $dt^2 + \varphi_j^2 \eta_i^2 + g_{\Sigma_i}$ the self-dual form $(dt \wedge \varphi_j \eta_i)^+$ acts by Clifford multiplication with eigenvalues $\pm i$. The eigenvalues of the operator P_j are then given by 0 and $-\chi_j \frac{\partial_t^2 \varphi_j}{\varphi_j}$.

Lemma 8.2. *There exists a constant $Q > 0$ such that*

$$|\psi_j(x)|^2 \leq Q$$

for every j and $x \in \overline{M}$.

Proof. Since

$$\Delta |\psi_j|^2 + 2 |\nabla_{A_j} \psi_j|^2 = 2 \operatorname{Re} \langle \nabla_{A_j}^* \nabla_{A_j} \psi_j, \psi_j \rangle$$

if x_j is a maximum point for $|\psi_j|^2$ we have $\operatorname{Re} \langle \nabla_{A_j}^* \nabla_{A_j} \psi_j, \psi_j \rangle \geq 0$. In conclusion

$$0 \geq \operatorname{Re} \langle \{P_j + P_j^b\} \psi_j, \psi_j \rangle_{x_j} + \frac{|\psi_j|_{x_j}^4}{4}.$$

By construction the operator $P_j + P_j^b$ is uniformly bounded from below, the proof is then complete. \square

Since $F_{A_j}^+ - F_{B_j}^+ = q(\psi_j)$ and by Lemma 8.2 the norms of the ψ_j are uniformly bounded, a similar estimate holds for $F_{A_j}^+ - F_{B_j}^+$.

Lemma 8.3. *There exists a constant $Q > 0$ such that*

$$\|\nabla_{A_j} \psi_j\|_{L^2(\overline{M}, g_j)} \leq Q$$

for any j .

Proof. We have

$$\begin{aligned}
0 &= \int_{\overline{M}} \operatorname{Re} \langle \nabla_{A_j}^* \nabla_{A_j} \psi_j, \psi_j \rangle d\mu_{g_j} + \int_{\overline{M}} \operatorname{Re} \langle \{P_j^b + P_j\} \psi_j, \psi_j \rangle d\mu_{g_j} \\
&\quad + \frac{1}{2} \int_{\overline{M}} \operatorname{Re} \langle q(\psi_j) \psi_j, \psi_j \rangle d\mu_{g_j} \\
&= \|\nabla_{A_j} \psi_j\|_{L^2(\overline{M}, g_j)}^2 + \int_{\overline{M}} \operatorname{Re} \langle \{P_j^b + P_j\} \psi_j, \psi_j \rangle d\mu_{g_j} + \frac{1}{4} \int_{\overline{M}} |\psi_j|^4 d\mu_{g_j}
\end{aligned}$$

but now

$$\int_{\overline{M}} \operatorname{Re} \langle \{P_j^b + P_j\} \psi_j, \psi_j \rangle d\mu_{g_j} \geq -k \|\psi_j\|_{L^2(\overline{M}, g_j)}^2$$

which then implies

$$\begin{aligned}
\|\nabla_{A_j} \psi_j\|_{L^2(\overline{M}, g_j)}^2 &\leq k \|\psi_j\|_{L^2(\overline{M}, g_j)}^2 - \frac{1}{4} \|\psi_j\|_{L^4(\overline{M}, g_j)}^4 \\
&\leq k \|\psi_j\|_{L^2(\overline{M}, g_j)}^2.
\end{aligned}$$

Since by Proposition 5.2 the volumes of the Riemannian manifolds (\overline{M}, g_j) are uniformly bounded, the lemma follows from Lemma 8.2. \square

Define $C_j = A_j - B_j$ and let C be a fixed smooth connection on the line bundle $L \otimes \mathcal{O}(-\Sigma)$. By the Hodge decomposition theorem we can write

$$C_j = C + \eta_j + \beta_j$$

where η_j is g_j -harmonic and $\beta_j \in (\mathcal{H}_{g_j}^1)^\perp$. Thus

$$F_{C_j}^+ = q(\psi_j) = F_C^+ + d^+ \beta_j.$$

Since C is a fixed connection 1-form, $\|F_C\|_{L^2(\overline{M}, g_j)}$ is uniformly bounded in the index j . As a result, there exists $Q > 0$ such that

$$\|d^+ \beta_j\|_{L^2(\overline{M}, g_j)} \leq Q$$

for any j . By the Stokes' theorem

$$\|d^+ \beta_j\|_{L^2(\overline{M}, g_j)}^2 = \|d^- \beta_j\|_{L^2(\overline{M}, g_j)}^2$$

and we then obtain an uniform upper bound on $\|d\beta_j\|_{L^2(\overline{M}, g_j)}$. By gauge fixing, see for example Section 5.3. in [25], we can always assume $d^* \beta_j = 0$ and α_j bounded in $H^1(\overline{M}, \mathbb{R})$. The Poincaré inequality given in Proposition 6.8 can then be used to conclude that

$$c \|\beta_j\|_{L^2(\overline{M}, g_j)}^2 \leq \|d\beta_j\|_{L^2(\overline{M}, g_j)}^2 \leq 2Q.$$

By a diagonal argument we can now extract a limit $\beta_j \rightarrow \beta$ with $\beta \in L_1^2(M, g)$. Similarly we extract a limit $\eta_j \rightarrow \eta$ with $\eta \in L^2(M, g)$ and harmonic with respect to g , see Proposition 6.5.

Define $a_j = \eta_j + \beta_j$ that by construction satisfies $d^*a_j = 0$. If we fix a compact set $K \subset M$, there exists j_0 such that for any $j \geq j_0$ the connection B_j restricted to K is trivial. Thus, for any $j \geq j_0$ we have $A_j = C_j$ and then $C = A_j - a_j$. We know that a_j is uniformly bounded in $L^2(\overline{M}, g_j)$, by using Lemma 8.3 we conclude that $\|\nabla_C \psi_j\|_{L^2(K, g_j)}^2$ is bounded independently of j . On this compact set K we can therefore extract a weak limit of the sequence $\{\psi_j\} \rightharpoonup \psi$. By using a diagonal argument and recalling that in a Hilbert space the norm is lower semicontinuous with respect the weak convergence, we obtain a limit $\psi \in L_1^2(M, g)$. Now, a bootstrap argument based on the *ellipticity* of the Seiberg-Witten equations can be used to conclude that the ψ_j are indeed smooth and that they converge, in the C^∞ -topology on compact sets, to ψ . By Lemma 8.2, ψ is uniformly bounded over M .

Let us summarize the discussion above into a theorem.

Theorem A. *Fix a $Spin^c$ structure on \overline{M} with determinant line bundle L . Let g be a metric on M asymptotic to a Poincaré metric in the C^2 -topology, and let $\{g_j\}$ the sequence of metrics on \overline{M} that approximate g . Let $\{(A_j, \psi_j)\}$ be the sequence of solutions of the SW equations with perturbations $\{F_{B_j}^+\}$ on $\{(\overline{M}, g_j)\}$. Then, up to gauge transformations, the solutions $\{(A_j, \psi_j)\}$ converge, in the C^∞ -topology on compact sets, to a solution (A, ψ) of the unperturbed SW equations on (M, g) such that*

- $A = C + a$ where C is a fixed smooth connection on $L \otimes \mathcal{O}(-\Sigma)$, $d^*a = 0$ and $a \in L_1^2(\Omega_g^1(M))$;
- $\psi \in L_1^2(M, g)$ and there exists $Q > 0$ such that $\sup_{x \in M} |\psi(x)| \leq Q$.

It remains to show that Theorem A can be successfully applied in the case of a minimal pair (\overline{M}, Σ) of logarithmic general type. Furthermore, we have to prove the solution (A, ψ) so constructed is irreducible.

Recall that by construction \overline{M} is a Kähler surface. Let us consider the standard $Spin^c$ structure on \overline{M} associated to the complex structure J . Let ω be a Kähler metric compatible with J , then on $(\overline{M}, \omega, J)$ it is easy to construct an irreducible solution of the perturbed SW equations. More precisely, $\overline{\psi} = (1, 0) \in \Omega^{0,0} \oplus \Omega^{0,2}$ and the Chern

connection \overline{A} on $K_{\overline{M}}^{-1}$ are solution of

$$\begin{cases} \mathcal{D}_{\overline{A}}\overline{\psi} = 0 \\ F_{\overline{A}}^+ + i(\frac{s_{\omega}+1}{4})\omega = q(\overline{\psi}) \end{cases}$$

where by s_{ω} we denote the scalar curvature of the Riemannian metric associated to ω . Now, if $b_2^+(\overline{M}) > 1$ a cobordism argument [25] allows us to conclude that the solutions (A_j, ψ_j) of 7 are indeed irreducible. On the other hand, when $b_2^+(\overline{M}) = 1$ we have to check that the cohomology classes represented by ω and $c_1(\mathcal{L})$ are in the same chamber. First, recall that the logarithmic canonical bundle \mathcal{L} associated to (\overline{M}, Σ) has positive self-intersection, see Proposition 2.2. Moreover, since the Kodaira dimension of \mathcal{L} is non-negative we have that for some integer m the divisor associated to $m\mathcal{L}$ is effective. We conclude that $\omega \cdot \mathcal{L} > 0$. It then follows that

$$(t\omega + (1-t)\mathcal{L})^2 > 0$$

for any $t \in [0, 1]$. The (A_j, ψ_j) are then irreducible. Finally, combining Proposition 2.2, Lemma 4.1 and Proposition 7.5, we get that the solution (A, ψ) given in Theorem A is irreducible.

9. APPLICATIONS

In a decade long effort, Claude LeBrun has brought to light a beautiful and deep connection between Seiberg-Witten theory and the Riemannian geometry of closed 4-manifolds; see for example [22], [24] and the bibliography therein. Many of these results still represent the cutting edge of our current understanding of Riemannian geometry in real dimension four.

In this section, we show how the analytical results previously derived in this work are robust enough to prove suitable generalizations of many of LeBrun's results.

Let us begin by reviewing Chern-Weil theory in the Poincaré metric. Recall that for a compact oriented 4-manifold N , the Gauss-Bonnet and Hirzebruch theorems state that

$$\chi(N) = \int_N E(g) d\mu_g, \quad \sigma(N) = \int_N L(g) d\mu_g$$

where $E(g)$ and $L(g)$ are respectively the Euler and signature forms associated to the metric g .

For non-compact manifolds the above curvature integrals might be not defined or dependent on the choice of the metric. Nevertheless, if the manifold has finite volume and bounded curvature these curvature

integrals are defined. In this case it remains to study their metric dependence. Here, we want to compute

$$\chi(M, g) = \int_M E(g) d\mu_g, \quad \sigma(M, g) = \int_M L(g) d\mu_g$$

when M is obtained from a pair (\overline{M}, Σ) of logarithmic general type and g is C^2 -asymptotic to a Poincaré metric at infinity. The following proposition computes the characteristic numbers of (M, g) in terms of $\chi(\overline{M})$, $\sigma(\overline{M})$, and a contribution coming from the cusp ends of M .

Proposition 9.1. *Let M be equipped with a metric g asymptotic in the C^2 -topology to a Poincaré metric. Then, we have the equalities*

$$\chi(M, g) = \chi(\overline{M}) - \chi(\Sigma) = \chi(M), \quad \sigma(M, g) = \sigma(\overline{M}) - \frac{1}{3}\Sigma^2,$$

where by Σ^2 we indicate the self-intersection of the boundary divisor.

Proof. A proof can be given generalizing a computation of Biquard, see Proposition 3.4. in [5]. Alternatively, one can apply a very general result of M. Stern, see Theorem 1.1 in [28]. \square

We can now study the Riemannian functional $\int_M s_g^2 d\mu_g$ restricted to the space of metrics with Poincaré type asymptotic.

Theorem B. *Let (\overline{M}, Σ) be a minimal pair of logarithmic general type. Assume the boundary divisor Σ to be smooth and without rational components. Let M be equipped with a metric g asymptotic to a Poincaré metric in the C^2 -topology. Then*

$$\frac{1}{32\pi^2} \int_M s_g^2 d\mu_g \geq (c_1^+(\mathcal{L}))^2$$

with equality if and only if g has constant negative scalar curvature, and is Kähler with respect to a complex structure compatible with the standard $Spin^c$ structure on M .

Proof. The first step is to apply Theorem A with respect to the standard $Spin^c$ structure on \overline{M} . The solution so obtained (A, ψ) is then irreducible and $\psi \in L_1^2$. Finally, a Bochner type argument as in Theorem 4.2 concludes the proof. \square

Regarding the equality case in Theorem B, H. Auvray has recently proved a *uniqueness* result, see [2]. More precisely, he is able to show the uniqueness of Kähler metrics, in arbitrary classes, with constant scalar curvature and Poincaré type asymptotic. Up to now, the result is proved for pairs (\overline{M}, Σ) with *ample* log-canonical bundle. In the

case of log-surfaces with smooth boundary, the ampleness of the log-canonical bundle can be neatly characterized, see Proposition 2.3.

Finally, Theorem B can be used to prove the following:

Corollary 9.2. *Let (\overline{M}, Σ) be as above. Then*

$$\frac{1}{32\pi^2} \int_M s_g^2 d\mu_g \geq (K_{\overline{M}} + \Sigma)^2$$

with equality iff the metric g is Kähler-Einstein with respect to a complex structure compatible with the standard $Spin^c$ structure on M .

In the last thirty years, the problem of constructing Kähler-Einstein metrics for pairs (\overline{M}, Σ) of log-general type was addressed by several authors, see for example [7], [14] and [29]. In [7], [14], a unique Kähler-Einstein metric is constructed for any pair (\overline{M}, Σ) with ample log-canonical bundle. This metric is *quasi-isometric* to a Poincaré type metric at infinity.

A more general existence and uniqueness theorem is proved by Tian-Yau in [29]. Nevertheless, the asymptotic behavior of the Kähler-Einstein metric so constructed is much more complicated to understand. In fact, it must be quite different from a Poincaré type metric as one can check on explicit examples coming from the theory of locally symmetric varieties.

We conclude this section by presenting an obstruction for Einstein metrics on blow-ups.

Theorem C. *Let (M, g) as above. Let M' be obtained from M by blowing up k points. If $k \geq \frac{2}{3}(K_{\overline{M}} + \Sigma)^2$, then M' does not admit a Poincaré type Einstein metric.*

Proof. By a result of Morgan-Friedman [11], we know that the manifold $\overline{M} \# k \mathbb{C}P^2$ admits at least 2^k different $Spin^c$ structures with determinant line bundles

$$L = K_{\overline{M}}^{-1} \pm E_1 \pm \dots \pm E_k$$

for which the SW equations have irreducible solutions for each metric. Since

$$\begin{aligned} (c_1(L)^+)^2 &= (c_1(\overline{M})^+ \pm E_1^+ \pm \dots \pm E_k^+)^2 \\ &= (c_1(\overline{M})^+)^2 + 2 \sum_i c_i(\overline{M})^+ \cdot \pm E_i^+ + \left(\sum_i \pm E_i^+ \right)^2 \end{aligned}$$

we can chose a $Spin^c$ structure whose determinant line bundle satisfies

$$(c_1(L)^+)^2 \geq (c_1(\overline{M})^+)^2 \geq c_1(\overline{M})^2 = c_1^2(\overline{M}).$$

We can now apply Theorem A for any of the $Spin^c$ structure above and with respect to the metric g on M' . We then construct 2^k irreducible solutions $(A, \psi) \in L_1^2(M', g)$, where $A = C + a$ with C a fixed smooth connection on $L \otimes \mathcal{O}(-\Sigma)$ and $a \in L_1^2(\Omega_g^1(M'))$. By appropriately choosing the $Spin^c$ structure and using Theorem 4.2 we compute

$$\begin{aligned} \frac{1}{32\pi^2} \int_{M'} s^2 d\mu_g &\geq (c_1(L \otimes \mathcal{O}(-\Sigma))^+)^2 \\ &\geq (K_{\overline{M}} + \Sigma)^2. \end{aligned}$$

By Proposition 9.1 one has

$$\begin{aligned} \chi(M', g) &= \chi(\overline{M}) - \chi(\Sigma) + k \\ \sigma(M', g) &= \sigma(\overline{M}) - \frac{1}{3}\Sigma^2 - k. \end{aligned}$$

Since $\chi(\Sigma) = -K_{\overline{M}} \cdot \Sigma - \Sigma^2$, if we assume g to be Einstein

$$\begin{aligned} (K_{\overline{M}} + \Sigma)^2 - k &= 2\chi(M') + 3\sigma(M') \\ &= \frac{1}{4\pi^2} \int_{M'} 2|W_+|^2 + \frac{s^2}{24} d\mu_g \\ &\geq \frac{1}{96\pi^2} \int_{M'} s^2 d\mu_g \\ &\geq \frac{1}{3} (K_{\overline{M}} + \Sigma)^2 \end{aligned}$$

so that

$$\frac{2}{3} (K_{\overline{M}} + \Sigma)^2 \geq k.$$

In other words if

$$k > \frac{2}{3} (K_{\overline{M}} + \Sigma)^2$$

we cannot have a Poincaré type Einstein metric on $M \# k \overline{\mathbb{C}P^2}$. The equality case can also be included and the proof goes as in the compact case. For more details, see [22]. The proof is then complete. \square

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